Math 4200
Friday September 18

1.6 branched domains (fundamental domains) to create single-valued analytic functions from multi-valued ones. (From Wednesday's notes).
2.1 begin Chapter 2 on definite and indefinite complex intergration

Chapter 2: Complex integration.
- Leads to Cauchy Integral Formula and magic theorems which result:
    - Liouville's Theorem: Bounded entire functions are constant.
    - Fundamental Theorem of Algebra: every degree n polynomial has n (complex) roots, counting multiplicity.
    - Magic ways to compute many definite integrals (contour integration).

<u>Announcements:</u>

2.1 Integration of complex-valued functions of a real variable t, just as in Calc 1. Introduction to *contour integrals* - analogous to *line integrals* from multivariable Calculus.

A1 Def: For $f: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ continuous, $f(t) = u(t) + i\, v(t)$, with $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$

$$\int_a^b f(t)\, \mathrm{d}t = \int_a^b u(t) + i\, v(t)\, \mathrm{d}t := \int_a^b u(t)\, \mathrm{d}t + i \int_a^b v(t)\, \mathrm{d}t.$$

It is useful for estimates to note that since u, v are continuous on $[a, b]$ they are uniformly continuous - and you proved in Math 3210 that in this case definite integrals are limits of Riemann sums for partionings P of $[a, b]$, as the "norm" of the partition approaches zero: For

$$P := \; a = t_0 < t_1 < \ldots \; < t_n = b$$
$$t_j \le t_j^* \; \le t_{j+t}, \qquad \Delta\, t_j = t_{j+1} - t_j$$
$$\|P\| := \max \Delta\, t_j\,,$$

$$\int_a^b u(t)\, \mathrm{d}t = \lim_{\|P\| \to 0} \sum_j u\left(t_j^*\right) \Delta t_j, \qquad \int_a^b v(t)\, \mathrm{d}t = \lim_{\|P\| \to 0} \sum_j v\left(t_j^*\right) \Delta t_j$$

so also

A2 Def:

$$\int_a^b f(t)\, \mathrm{d}t = \lim_{\|P\| \to 0} \sum_j u\left(t_j^*\right) \Delta t_j + i \lim_{\|P\| \to 0} \sum_j v\left(t_j^*\right) \Delta t_j = \lim_{\|P\| \to 0} \sum_j f\left(t_j^*\right) \Delta t_j.$$

Example 1: Use Calc 1 FTC to compute

$$\int_0^{\frac{\pi}{2}} -2 \sin(t) \cos(t) + i\left(\cos^2(t) - \sin^2(t)\right) \mathrm{d}t.$$

Use *A2* and the triangle inequality on Riemann sums to prove the important integral estimate which bounds the modulus of definite integrals in terms of the integrals of their modulus:

A3 <u>Theorem</u>

$$\left| \int_a^b f(t)\, \mathrm{d}t \right| \leq \int_a^b |f(t)| \mathrm{d}t.$$

A4 Fundamental Theorem of Calculus for $f : [a, b] \rightarrow \mathbb{C}$: Let $u, v : [a, b] \rightarrow \mathbb{R}$ continuous, $f(t) = u(t) + i\, v(t)$, $F(t)$ such that $F'(t) = f(t)$. Then

$$\int_a^b f(t)\, \mathrm{d}t = F(b) - F(a).$$

B1 Def Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then the *complex line integral* or *contour integral*

$$\int_\gamma f(z)\, \mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\, \mathrm{d}t$$

where we use the definition *A1* on the previous page to compute the integral on the right. Note, we have substituted $z = \gamma(t)$ and used the differential substitution, $dz = \gamma'(t)dt$ into the integrand.

B2 In the case that $\gamma'(t) \neq 0$ for any t it follows from the continuity of $|\gamma'(t)|$ that $|\gamma'(t)| \geq \delta > 0$ on $[a, b]$. And in this case the complex line integral above can be realized as a limit which explains the geometry of what's going on:

$$\int_\gamma f(z)\, \mathrm{d}z = \lim_{\max\{|\Delta z_j|\} \to 0} \sum_{j=0}^{n-1} f(z_j)\Delta z_j \ ,$$

$$P := \ a = t_0 < t_1 < \ldots < t_n = b;$$

$$\Delta t_j = t_{j+1} - t_j \qquad \|P\| := \max \Delta t_j$$

$$z_j = \gamma(t_j), \quad \Delta z_j = z_{j+1} - z_j = \gamma(t_{j+1}) - \gamma(t_j).$$

The reason this is true is that by the 3210 or 3220 affine approximation formula for the C^1 curve γ,

$$\gamma(t_{j+1}) - \gamma(t_j) = \gamma'(t_j)\Delta t_j \ + \varepsilon(t_j)\Delta t_j$$

where one can show that the $|\varepsilon(t)| \to 0$ uniformly as $\|P\| \to 0$ because γ is continuously differentiable. Also, because $M \geq |\gamma'(t)| \geq \delta$ the condition that $\max\{|\Delta z_j|\} \to 0$ in $\mathbb{C}$is equivalent to the $\|P\| \to 0$ in $[a, b]$, also because of the approximation formula. So,

$$\lim_{\max\{|\Delta z_j|\} \to 0} \sum_{j=0}^{n-1} f(z_j)\Delta z_j$$

$$= \lim_{\|P\|\to 0} \sum_j f(\gamma(t_j))\,(\gamma(t_{j+1}) - \gamma(t_j))$$

$$= \lim_{\|P\|\to 0} \sum_j f(\gamma(t_j))\,(\gamma'(t_j)\Delta t_j + \varepsilon(t_j)\Delta t_j)$$

$$= \lim_{\|P\|\to 0} \sum_j f(\gamma(t_j))\,\gamma'(t_j)\Delta t_j$$

$$= \int_a^b f(\gamma(t))\gamma'(t)\, \mathrm{d}t\ .$$

Example 2: Let $\gamma(t) = e^{i\,t}$, $0 \le t \le \frac{\pi}{2}$, $f(z) = z$. Compute

$$\int_{\gamma} f(z)\, dz.$$

Sketch. Do you think you would get the same answer if you followed the same quarter circle in the same direction, but with a different parameterization? What if you reversed direction? Could you explain why?

B3 Integral estimate: Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ continuous, $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then

$$\left| \int_\gamma f(z)\, \mathrm{d}z \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\, \mathrm{d}t \right|$$

$$\leq \int_a^b |f(\gamma(t))\gamma'(t)|\, \mathrm{d}t \qquad (A3)$$

$$= \int_a^b |f(\gamma(t))|\, |\gamma'(t)|\, dt$$

Def Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ continuous, $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then

$$\int_\gamma |f(z)|\, |\mathrm{d}z| := \int_a^b |f(\gamma(t))|\, |\gamma'(t)|\, dt$$

Using the definition, we see that the shorthand for the integral estimate in *B3* is

$$\left| \int_\gamma f(z)\, \mathrm{d}z \right| \leq \int_\gamma |f(z)|\, |\mathrm{d}z|.$$

Note that $|dz| = |\gamma'(t)|\, dt$ is the element of arclength.

Example 3: In Example 2, you showed that for $\gamma(t) = \mathrm{e}^{it},\ 0 \leq t \leq \frac{\pi}{2}$,

$$\int_\gamma z\, \mathrm{d}z = -1.$$

Compute

$$\int_\gamma |z|\, |\mathrm{d}z|$$

and verify the integral estimate *B3*. To be continued....

Math 4200-001
Week 4-5 concepts and homework
1.6, 2.1-2.2
Due Friday September 25 at 11:59 p.m.

1.6 10, 14
2.1 2ac, 3, 5, 10, 11, 13, 14;
2.2 1ad, 2 (prove with FTC!), 3 (work in reverse to rewrite as a contour integral that you can evaluate), 4, 6, 8, 9, 10 (use the antiderivative theorem and slightly modify Example 1.6.8).

Hint: In many of these problems the fundamental theorem of Calculus for contour integrals lets you find the answer very quickly once you find an antiderivative on an appropriate domain.

w4.1 (extra credit) This is a careful version of 1.6.6. Part (a) is relatively straightforward. I consider part (b) to be challenging.
a) Solve $\sin(z) = w$ for z using the quadratic formula and logarithms. Keep careful track of the multi-valued nature of the inverse sine function $\arcsin(z)$. Note that the quadratic formula yields two solutions except when $\cos(z) = 0$.
b) Prove that there is a branch of $\arcsin(z)$ defined on the branch domain we used in class for $\sqrt{z^2 - 1}$, namely $\mathbb{C} \setminus \{x \in \mathbb{R} \text{ s.t. } |x| \geq 1\}$, which is a bijection to the vertical strip $\left\{x + i\,y \,\middle|\, -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$. This branch extends the Calculus $\arcsin(x)$ which was defined as a differentiable function on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.